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Complete determination of the singularity structure of zeta functions

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Abstract. Series of extended Epstein type provide examples of non-trivial zeta functions with important physical applications. The regular part of their analytic continuation is seen to be a convergent or an asymptotic series. Their singularity structure is completely determined in terms of the Wodzicki residue in its generalized form, which is proven to yield the residua of all the poles of the zeta function, and not just that of the rightmost pole (obtainable from the Dixmier trace). The calculation is a very down-to-earth application of these powerful functional analytical methods in physics.

1. Introduction

A most important issue in the application of the zeta-function regularization method [1] in physics is the precise determination of the pole structre of the analytical continuation of the corresponding zeta function. In the recent mathematical literature, there are precise results which characterize the meromorphic structure of the analytical continuation of the zeta function of any elliptic pseudo-differential operator (ΨDO) [2], even of complex order [3]. The position and the order of the poles is known, and also the residue of the rightmost one, which can be determined by using either the Dixmier trace or the Wodzicki residue of the principal symbol of the operator. Here, we obtain the residua of all the remaining poles and illustrate their very simple calculation through such powerful functional analytical tools, by means of two fundamental examples with physical application [4]. The additional determination that is carried out of the regular part of the analytic continuation completes the analysis of the meromorphic structure of the zeta functions.

A pseudo-differential operator A of order m on a manifold M_n is defined through its symbol $a(x,\xi)$, which is a function belonging to the space $S^m(\mathbb{R}^n\times\mathbb{R}^n)$ of $\mathbb{C}^\infty$ functions such that for any α,β there exists a constant $C_{\alpha,\beta}$ so that

$$|\partial_\xi^\alpha\partial_x^\beta a(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}. \tag{1}$$

The definition of A is given (in the distribution sense) by

$$Af(x) = (2\pi)^{-n}\int \mathrm{e}^{\mathrm{i}\langle x,\xi\rangle} a(x,\xi)\hat{f}(\xi)\,\mathrm{d}\xi \tag{2}$$

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where f is a smooth function ($f \in \mathcal{S}$) and $\hat{f}$ its Fourier transform. When $a(x, \xi)$ is a polynomial in ξ one gets a differential operator but, in general, the order m can be even complex. Pseudo-differential operators are useful tools, both in mathematics and in physics. They were crucial for the proof of the uniqueness of the Cauchy problem [5] and also for the proof of the Atiyah–Singer index formula [6]. In quantum field theory they appear in any analytic continuation process (as complex powers of differential operators, like the Laplacian) [7]. They currently constitute the basic starting point of any rigorous formulation of quantum field theory through microlocalization [8], a concept that is considered to be the most important step towards the understanding of linear partial differential equations since the appearance of distributions.

For A a positive-definite elliptic ΨDO of positive order $m \in \mathbb{R}$, acting on the space of smooth sections of an n-dimensional vector bundle E over a closed, n-dimensional manifold M, the zeta function is defined as

$$\zeta_A(s) = \operatorname{tr} A^{-s} = \sum_j \lambda_j^{-s} \qquad \operatorname{Re} s > \frac{n}{m} \equiv s_0. \tag{3}$$

Here s_0 is called the abscissa of convergence of $\zeta_A(s)$, which is proven to have a meromorphic continuation to the whole complex plane $\mathbb{C}$ (regular at s_0), provided that A admits a spectral cut: $L_\theta = \{\lambda \in \mathbb{C}; \operatorname{Arg} \lambda = \theta, \theta_1 < \theta < \theta_2\}$, $\operatorname{Spec} A \cap L_\theta = \emptyset$ (the Agmon–Nirenberg condition). Strictly speaking, the definition of $\zeta_A(s)$ depends on the position of the cut L_θ, not so that of the determinant [9] $\det_\zeta A = \exp[-A'(0)]$, which only depends on the homotopy class of the cut. The precise structure of the analytical continuation is well known [10]; it has at most simple poles at

$$s_k = (n-k)/m \qquad k = 0, 1, 2, \ldots, n-1, n+1, \ldots. \tag{4}$$

The applications of this zeta-function definition of a determinant in physics are important [11, 12]. A zeta function with the same meromorphic structure in the complex s-plane and extending the ordinary definition to operators of complex order $m \in \mathbb{C}\backslash\mathbb{Z}$, has been recently obtained in [3]. (It is clear that operators of complex order do not admit spectral cuts.) The construction in [3] starts from the definition of a trace, obtained as the integral over the manifold of the trace density of the difference between the Schwartz kernel of A and the Fourier transform of a number of first homogeneous terms (in ξ) of the usual decomposition of the symbol of A:

$$a(x, \xi) = a_m(x, \xi) + a_{m-1}(x, \xi) + \cdots + a_{m-N}(x, \xi) + \cdots. \tag{5}$$

2. The Dixmier trace and the Wodzicki residue

In order to write down an action in operator language one needs a functional that replaces integration. For the Yang–Mills theory this is the Dixmier trace, which constitutes the unique extension of the usual trace to the ideal $\mathcal{L}^{(1,\infty)}$ of the compact operators T such that the partial sums of its spectrum diverge logarithmically as the number of terms in the sum, i.e.

$$\sigma_N(T) \equiv \sum_{j=0}^{N-1} \mu_j = \mathcal{O}(\log N) \qquad \mu_0 \geqslant \mu_1 \geqslant \cdots. \tag{6}$$

The definition of the Dixmier trace of T, Dtr T [13], is then a refinement of the limit

$$\lim_{N\to\infty} \frac{1}{\log N} \sigma_N(T). \tag{7}$$

(It is directly given by this limit when the Cesaro means $M(\rho)$ of the sequence in N are convergent as $\rho \to \infty$.) As observed by Connes [14], the Hardy–Littlewood theorem can be stated in a way that connects the Dixmier trace with the residue of the zeta function of the operator T^{-1} at $s = 1$:

$$\mathrm{Dtr}\, T = \lim_{s\to 1^+} (s-1)\zeta_{T^{-1}}(s). \tag{8}$$

The Wodzicki (or non-commutative) residue [15] is the only extension of the Dixmier trace to ΨDOs which are not in $\mathcal{L}^{(1,\infty)}$. Even more, it is the only trace one can define in the algebra of ΨDOs up to a multiplicative constant. It is given by the integral

$$\mathrm{res}\, A = \int_{S^*M} \mathrm{tr}\, a_{-n}(x,\xi)\, \mathrm{d}\xi \tag{9}$$

with $S^*M \subset T^*M$ the co-sphere bundle on M (some authors put a coefficient in front of the integral). If $\dim M = n = -\mathrm{ord}\, A$ (M compact Riemann, A elliptic, $n \in \mathbb{N}$) it coincides with the Dixmier trace, and one has [15]

$$\mathrm{Res}_{s=1}\zeta_A(s) = \frac{1}{n}\mathrm{res}\, A^{-1}. \tag{10}$$

However, the Wodzicki residue continues to make sense for ΨDOs of arbitrary order and, even if the symbols $a_j(x,\xi)$, $j < m$, are not invariant under coordinate choice, the integral in (9) is, and defines a trace. In particular, the residua of the poles of the extended definition of the zeta function to operators of complex order are also given by the non-commutative residue. Moreover, an interesting connection of the Wodzicki residue with the second coefficient of the heat-kernel expansion of the Laplacian has recently been found [16, 17]. An overview on physical applications of noncommutative geometry can be found in [18] (see also references therein).

The advantage of the explicit computation of the residues of the poles of the zeta function by this method—which relies in its extreme simplicity—will become clear when we discuss some basic examples in the following section.

3. Calculation of the residues of the poles of the zeta function

A complete determination of the meromorphic structure of the zeta function in the complex plane is obtained as follows. Relying on the above results, what is missing for the description of the singularities are the residua of all the remaining poles. As for the regular part of the analytic continuation, specific methods have to be used and the results are non-trivial; asymptotic series and not convergent ones, appear most often [19].

Proposition 1. Under the conditions of existence of the zeta function of A, given above, and assuming that the symbol $a(x,\xi)$ of the operator A is analytic in ξ^{-1} at $\xi^{-1} = 0$, the formula for the determination of the residue of the rightmost pole (by means of the Wodzicki residue) can be generalized to calculate *all* the residua of the zeta function poles, in the way:

$$\mathrm{Res}_{s=s_k}\zeta_A(s) = \frac{1}{m}\mathrm{res}\, A^{-s_k} = \frac{1}{m}\int_{S^*M} \mathrm{tr}\, a_{-n}^{-s_k}(x,\xi)\, \mathrm{d}^{n-1}\xi. \tag{11}$$

Proof. One just has to notice that the homogeneous component of degree $-n$ of the corresponding power of the principal symbol of A is obtained by taking the appropriate derivative of a power of the symbol with respect to ξ^{-1} at $\xi^{-1} = 0$, namely

$$a_{-n}^{-s_k}(x,\xi) = \left(\frac{\partial}{\partial \xi^{-1}}\right)^k [\xi^{n-k} a^{(k-n)/m}(x,\xi)]\Big|_{\xi^{-1}=0} \xi^{-n}. \tag{12}$$

The proof then follows by simple algebraic manipulation. □

These results will now be illustrated with two examples. Aside from the Riemann and Hurwitz zeta functions, the Epstein ones [20] and generalizations thereof are most basic tools in the zeta-function regularization method [19]. They appear in the calculation of the vacuum energy or effective potentials of quantum physical systems involving toroidal compactification, finite temperature, massive particles, or a chemical potential. Consider a spacetime with topology $\mathbb{R} \times T^2$ [21] and a general metric on T^2: $\mathrm{d}s^2 = h_{ab}\,\mathrm{d}x^a\,\mathrm{d}x^b$, with

$$h_{ab} = \frac{1}{\tau_2}\begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix} \tag{13}$$

(τ_1, τ_2) being the Teichmüller parameters, $\tau = \tau_1 + \mathrm{i}\tau_2$, $\tau_2 > 0$. The Laplace–Beltrami operator is: $L = -\tau_2^{-1}(|\tau|^2\partial_1^2 - 2\tau_1\partial_1\partial_2 + \partial_2^2)$ and its eigenvalues $\lambda_{n_1,n_2} = 4\pi^2\tau_2^{-1}(|\tau|^2 n_1^2 - 2\tau_1 n_1 n_2 + n_2^2)$. In the massive case the spectrum runs over $n_1, n_2 \in \mathbb{Z}$. If $m = 0$ the zero-mode $n_1 = n_2 = 0$ has to be excluded. Under different boundary conditions (Dirichlet and Neumann, for instance) one gets a restriction of the indices to non-negative values and in many situations—as is the case of spherical compactification—a one-dimensional variant of the Laplacian appears [19]. This leads us to consider two families of such operators—plus boundary conditions in general. The corresponding zeta functions belong to the family of generalized Epstein zeta functions

$$\zeta_E(s; a, b, c; q) \equiv \sum_{m,n\in\mathbb{Z}} (am^2 + bmn + cn^2 + q)^{-s} \qquad \mathrm{Re}\, s > 1 \tag{14}$$

(where q is the mass, chemical potential or finite-temperature contribution) or to the simpler version [22]

$$\zeta_G(s; a, c; q) \equiv \sum_{n=-\infty}^{\infty} [a(n+c)^2 + q]^{-s} \qquad \mathrm{Re}\, s > 1/2. \tag{15}$$

The restriction of these series to non-negative values of the indices will be denoted by ζ_{E_t} and ζ_{G_t}, respectively. The parenthesis in (14) is an inhomogeneous quadratic form, $Q(x, y) + q$, restricted to the integers. We assume that $a, c > 0$ and $\Delta = 4ac - b^2 > 0$ [23]. The starting point for the derivation of the formulae is Jacobi's theta function identity

$$\sum_{n=-\infty}^{\infty} \mathrm{e}^{-(n+z)^2 t} = \sqrt{\frac{\pi}{t}}\left[1 + \sum_{n=1}^{\infty} \mathrm{e}^{-\pi^2 n^2/t}\cos(2\pi n z)\right] \qquad z, t \in \mathbb{C}, \mathrm{Re}\, t > 0. \tag{16}$$

Example 1. Consider first the second function, which is simpler. Making use of the Jacobi identity we get the analytical continuation

$$\zeta_G(s; a, c; q) = \sqrt{\frac{\pi}{a}}\frac{\Gamma(s-1/2)}{\Gamma(s)} q^{1/2-s} + \frac{4\pi^s}{\Gamma(s)} a^{-1/4-s/2} q^{1/4-s/2}$$
$$\times \sum_{n=1}^{\infty} n^{s-1/2}\cos(2\pi n c) K_{s-1/2}(2\pi n\sqrt{q/a}) \tag{17}$$

where K_ν is the modified Bessel function of the second kind. Associated with the above zeta functions, but considerably more difficult to treat, are the corresponding truncated sums, with indices running from 0 to ∞. In this case the Jacobi identity is of no use. By means of specific techniques of analytic continuation of zeta functions [19], we obtain

$$\zeta_{G_t}(s;a,c;q) \sim \left(\frac{1}{2}-c\right)q^{-s} + \frac{q^{-s}}{\Gamma(s)}\sum_{m=1}^{\infty}\frac{(-1)^m\Gamma(m+s)}{m!}\left(\frac{q}{a}\right)^{-m}\zeta_H(-2m,c)$$
$$+\sqrt{\frac{\pi}{a}}\,\frac{\Gamma(s-1/2)}{2\Gamma(s)}q^{1/2-s} + \frac{2\pi^s}{\Gamma(s)}a^{-1/4-s/2}q^{1/4-s/2}$$
$$\times\sum_{n=1}^{\infty}n^{s-1/2}\cos(2\pi nc)K_{s-1/2}(2\pi n\sqrt{q/a}). \tag{18}$$

The first series is asymptotic [23, 24]. From the previous expressions one can calculate the determinants of Klein–Gordon and Dirac operators on compact spaces as, for instance, N-cubes, cylinders and spheres S^N (whenever the spectrum λ_n is a polynomial in n).

The meromorphic structure of these zeta functions is described by the general theory. According to it, in principle, poles at the positions $s = -1, -2, -3, \ldots$ could also be possible. They just have zero residue, as we shall now prove. The residue of the rightmost pole at $s = 1/2$ can be obtained from the Dixmier trace:

$$\operatorname{Dtr} G^{-1/2} = \lim_{N\to\infty}\frac{1}{\log N}\sum_{j=0}^{N-1}[a(j+c)^2+q]^{-1/2} = \frac{1}{\sqrt{a}} \tag{19}$$

which is in fact the value of the residue of the pole at $s = 1/2$ in (17), thus $\operatorname{Res}_{s=1/2}\zeta_G(s) = \operatorname{Dtr} G^{-1/2}$. Now to the second step: the residue of this pole at $s = 1/2$ can also be obtained from the Wodzicki residue. In fact, we have:

$$\operatorname{res} G^{-1/2} = \int_{S^*R}\operatorname{tr} g_{-1}^{-1/2}(\xi)\,\mathrm{d}\xi = \frac{2}{\sqrt{a}} \tag{20}$$

(note that $g^{-1/2}(\xi) = \xi^{-1} + \mathcal{O}(\xi^{-2})$ and that the zero-dimensional sphere is reduced to two points, namely $S^*R = S^0 = \{-1, 1\}$). Thus,

$$\operatorname{Res}_{s=1/2}\zeta_G(s) = \tfrac{1}{2}\operatorname{res} G^{-1/2}. \tag{21}$$

Having dealt with the rightmost pole in all possible ways, we now analyse the others by means of the generalized Wodzicki residue. We will prove that

$$\operatorname{Res}_{s=1/2-k}\zeta_G(s) = \tfrac{1}{2}\operatorname{res} G^{k-1/2} \qquad \operatorname{Res}_{s=-k}\zeta_G(s) = 0 \qquad k = 0, 1, 2, \ldots. \tag{22}$$

The decomposition of the corresponding symbol into homogeneous parts yields

$$g^{k-1/2}(\xi) = \xi^{2k-1}\left[1 + \frac{k-1/2}{1!}\frac{q}{\xi^2} + \cdots + \frac{(k-1/2)(k-3/2)\ldots 1/2}{k!}\frac{q^k}{\xi^{2k}} + \cdots\right]$$
$$= \xi^{2k-1} + \cdots + \frac{(2k-1)!!}{k!2^k}\frac{q^k}{\xi} + \cdots \qquad k = 0, 1, 2, \ldots$$
$$g^k(\xi) = (\xi^2+q)^k = \xi^{2k} + \cdots + q^k \qquad k = 0, 1, 2, \ldots \tag{23}$$

therefore,

$$\operatorname{tr} g_{-1}^{k-1/2}(\xi) = \frac{(2k-1)!!q^k}{k!2^k} \qquad \operatorname{tr} g_{-1}^k(\xi) = 0 \qquad k = 0, 1, 2, \ldots. \tag{24}$$

Thus, we obtain

$$\mathrm{Res}_{s=1/2-k}\zeta_G(s) = \frac{1}{2}\,\mathrm{res}\,G^{k-1/2} = \frac{(2k-1)!!q^k}{k!2^k}$$
$$\mathrm{Res}_{s=-k}\zeta_G(s) = \frac{1}{2}\,\mathrm{res}\,G^{k} = 0 \qquad k = 0, 1, 2, \ldots \tag{25}$$

which coincide in fact with the residues of the poles of $\zeta_G(s)$ at $s = s_k$, as we wanted to see. The fact that some of the would-be poles are actually not present follows from the decomposition (23) showing clearly that their residua are zero. To be noticed is the fact that the absence of these poles is a consequence of the well-known Seeley theorem, which is thus here obtained, in some way, as a bonus. We see in this example how the Wodzicki residue under its more general form allows us to calculate all the residua of the poles of the zeta function. The meromorphic structure of the analytical continuation of the zeta function is absolutely specified through the Dixmier trace and the Wodzicki residue in its general form. However, we have also shown through our explicit calculation that what remains can in no way be considered as a trivial analytic part. It may be given by a convergent series but, possibly, by an asymptotic one.

Example 2. It is more involved. In the homogeneous case the analytic continuation of this Epstein zeta function is given by the Chowla–Selberg formula [23]

$$\zeta_E(s; a, b, c; 0) = 2\zeta(2s)a^{-s} + \frac{2^{2s}\sqrt{\pi}\,a^{s-1}}{\Gamma(s)\Delta^{s-1/2}}\Gamma(s-1/2)\zeta(2s-1)$$
$$+\frac{2^{s+5/2}\pi^s}{\Gamma(s)\Delta^{s/2-1/4}\sqrt{a}}\sum_{n=1}^{\infty} n^{s-1/2}\sigma_{1-2s}(n)\cos(\pi n b/a)K_{s-1/2}\left(\frac{\pi n}{a}\sqrt{\Delta}\right). \tag{26}$$

where $\sigma_s(n) \equiv \sum_{d|n} d^s$, sum over the s-powers of the divisors of n. (There is a misprint in the transcription of formula (26) in [26].) We observe that the right-hand side of (26) exhibits a simple pole at $s = 1$, with residue

$$\mathrm{Res}_{s=1}\zeta_E(s; a, b, c; 0) = \frac{2\pi}{\sqrt{\Delta}}. \tag{27}$$

In the general case, we have obtained the meromorphic continuation

$$\zeta_E(s; a, b, c; q) = \frac{2\pi q^{1-s}}{(s-1)\sqrt{\Delta}} + \frac{4}{\Gamma(s)}\Bigg[\left(\frac{q}{a}\right)^{1/4}\left(\frac{\pi}{\sqrt{qa}}\right)^{s}$$
$$\times\sum_{k=1}^{\infty} k^{s-1/2}K_{s-1/2}\left(2\pi k\sqrt{\frac{q}{a}}\right) + \sqrt{\frac{q}{a}}\left(2\pi\sqrt{\frac{a}{q\Delta}}\right)^{s}\sum_{k=1}^{\infty} k^{s-1}K_{s-1}\left(4\pi k\sqrt{\frac{aq}{\Delta}}\right)$$
$$+\sqrt{\frac{2}{a}}(2\pi)^s\sum_{k=1}^{\infty} k^{s-1/2}\cos(\pi k b/a)\sum_{d|k} d^{1-2s}\left(\Delta + \frac{4aq}{d^2}\right)^{1/4-s/2}$$
$$\times K_{s-1/2}\left(\frac{\pi k}{a}\sqrt{\Delta + \frac{4aq}{d^2}}\right)\Bigg]. \tag{28}$$

This is a fundamental result. It looks rather different from the Chowla–Selberg formula (26), but it can actually be viewed as its natural extension to the case $q \neq 0$. It also shares all the good properties of (26).

As in example 1, the only pole of this zeta function can be obtained by using either the Dixmier trace or the Wodzicki residue. In fact,

$$\mathrm{Dtr}\, E^{-1} = \lim_{N\to\infty}\frac{1}{\log(N^2)}\sum_{m,n=-N}^{N}(am^2 + bmn + cn^2 + q)^{-1} = \frac{2\pi}{\sqrt{\Delta}} \tag{29}$$

which is the value of the residue of the pole at $s = 1$, thus

$$\mathrm{Res}_{s=1}\zeta_E(s) = \mathrm{Dtr}\, E^{-1}. \tag{30}$$

Moreover, this value can also be obtained from the Wodzicki residue

$$\mathrm{res}\, E^{-1} = \int_{S^*\mathbb{R}^2} \mathrm{tr}\, e_{-2}^{-1}(\xi)\, \mathrm{d}^2\xi = \frac{1}{a}\int_0^{2\pi} \left[\left(\tan\theta + \frac{b}{2a}\right)^2 + \frac{\Delta}{4a^2}\right]^{-1} \mathrm{d}(\tan\theta) = \frac{4\pi}{\sqrt{\Delta}}. \tag{31}$$

Here the integral is performed over the unit circumference ($S^*\mathbb{R}^2 = S^1$, $|\xi| = 1$). Thus,

$$\mathrm{Res}_{s=1}\zeta_E(s) = \tfrac{1}{2}\, \mathrm{res}\, E^{-1}. \tag{32}$$

We have thus shown again how the rightmost pole of the zeta function can be obtained either from the Dixmier trace or from the Wodzicki residue. The fact that this is the only pole of our zeta function also follows from the calculation of the generalized Wodzicki residua. According to the general theory, the other possible poles would be at $s = s_k = 1 - k/2$, $k = 1, 3, 4, \ldots$. We must obtain the homogeneous component of degree -2 of the principal symbol of the operator $E^{k/2-1}$

$$e^{k/2-1}(\xi_1, \xi_2) = (a\xi_1^2 + b\xi_1\xi_2 + c\xi_2^2 + q)^{k/2-1}. \tag{33}$$

But it is clear that neither for k odd nor for k even is there any component of this principal symbol of degree -2. All corresponding residua are zero and none of these poles exists.

The most difficult case in the family of Epstein-like zeta functions corresponds to having a *truncated* range. This comes about when one imposes boundary conditions of the usual Dirichlet or Neumann type [19]. Jacobi's theta-function identity is then useless and no expression in terms of a convergent series for the analytical continuation to values of Re s below the abscissa of convergence can be obtained. The best one gets is an asymptotic series expression. However, the issue of extending the Chowla–Selberg formula or, better still, the more general one we have obtained before for inhomogeneous Epstein zeta functions in two indices, is not simple and has never been adressed in the literature. In order to obtain the analytic continuation to Re $s \leqslant 1$ of the truncated inhomogeneous Epstein zeta function in two dimensions,

$$\zeta_{E_t}(s; a, b, c; q) \equiv \sum_{m,n=0}^{\infty} (am^2 + bmn + cn^2 + q)^{-s} \tag{34}$$

we can proceed in two ways: either by a direct calculation that leads to the generalized Chowla–Selberg formula [19] or by using the formulae for the Epstein zeta function in one dimension (example 1) recurrently. In both cases the result is

$$\begin{aligned}
\zeta_{E_t}(s; a, b, c; q) &\equiv \sum_{m,n=0}^{\infty} (am^2 + bmn + cn^2 + q)^{-s} \\
&\sim \frac{(4a)^s}{\Gamma(s)} \sum_{m,n=1}^{\infty} \frac{(-1)^m \Gamma(m+s)}{m!} (2a)^{2m} (\Delta n^2 + 4aq)^{-m-s} \zeta_H\left(-2m; \frac{bn}{2a}\right) \\
&\quad - \frac{bq^{1-s}}{(s-1)\Delta\Gamma(s-1)} \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(n+s-1) B_n}{n!} \left(\frac{4aq}{\Delta}\right)^{-n} \\
&\quad + \frac{q^{-s}}{4} + \frac{\pi q^{1-s}}{2(s-1)\sqrt{\Delta}} + \frac{1}{4}\left(\sqrt{\frac{\pi}{a}} + \sqrt{\frac{\pi}{c}}\right) \frac{\Gamma(s-1/2)}{\Gamma(s)} q^{1/2-s} \\
&\quad + \frac{1}{\Gamma(s)} \left[2\left(\frac{q}{a}\right)^{1/4} \left(\frac{\pi}{\sqrt{qa}}\right)^s \sum_{k=1}^{\infty} k^{s-1/2} K_{s-1/2}\left(2\pi k \sqrt{\frac{q}{a}}\right)\right.
\end{aligned}$$

$$
+\left(\frac{4aq}{\Delta}\right)^{1/4}\left(2\pi\sqrt{\frac{a}{q\Delta}}\right)^{s}\sum_{k=1}^{\infty}k^{s-1/2}K_{s-1/2}\left(4\pi k\sqrt{\frac{aq}{\Delta}}\right)+\sqrt{\frac{q}{a}}\left(2\pi\sqrt{\frac{a}{q\Delta}}\right)^{s}
\times\sum_{k=1}^{\infty}k^{s-1}K_{s-1}\left(4\pi k\sqrt{\frac{aq}{\Delta}}\right)+\sqrt{\frac{2}{a}}(2\pi)^{s}\sum_{k=1}^{\infty}k^{s-1/2}\cos(\pi kb/a)
\times\sum_{d|k}d^{1-2s}\left(\Delta+\frac{4aq}{d^2}\right)^{1/4-s/2}K_{s-1/2}\left(\frac{\pi k}{a}\sqrt{\Delta+\frac{4aq}{d^2}}\right)\Bigg]. \qquad (35)
$$

This imposing formula is new too. The first series is in general asymptotic, but it converges for a wide range of values of the parameters. The second series is always asymptotic and contributes to the pole at $s = 1$. As in the case of the fundamental formula, (28), the pole structure is here explicitly, although much more elaborate. Apart from the pole at $s = 1$, there is here a sequence of poles at $s = \pm 1/2, -3/2, -5/2, \ldots$. Calculations similar to the previous ones lead to the determination of the residua of the poles in this case by means of the same expressions as before.

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References

[1] Hawking S W 1977 *Commun. Math Phys.* **55** 133
Dowker J S and Critchley R 1976 *Phys. Rev.* D **13** 3224
Brown L S and MacLay G J 1969 *Phys. Rev.* **184** 1272
[2] Hörmander L 1983–85 *The Analysis of Partial Differential Operators* vols I–IV (Berlin: Springer)
Treves F 1980 *Introduction to Pseudodifferential and Fourier Integral Operators* vols I and II (New York: Plenum)
Taylor M E 1981 *Pseudodifferential Operators* (Princeton, NJ: Princeton University Press)
Lawson H and Michelsohn M L 1989 *Spin Geometry* (Princeton, NJ: Princeton University Press)
[3] Kontsevich M and Vishik S 1993 Geometry of determinants of elliptic operators *Functional Analysis on the Eve of the 21st Century* vol 1, pp 137–97
[4] Elizalde E, Odintsov S D, Romeo A, Bytsenko A A and Zerbini S 1994 *Zeta Regularization Techniques with Applications* (Singapore: World Scientific)
[5] Calderón A P and Zygmund A 1957 *Am. J. Math.* **79** 801
Calderón A P and Zygmund A 1961 *Studia Math.* **20** 171
Calderón A P and Vaillancourt R 1972 *Proc. Nat. Acad. Sci. USA* **69** 1185
[6] Atiyah M and Singer I M 1968 *Ann. Math.* **87** 484
Atiyah M and Singer I M 1968 *Ann. Math.* **87** 546
Atiyah M and Singer I M 1971 *Ann. Math.* **93** 119
Atiyah M and Singer I M 1971 *Ann. Math.* **93** 139
[7] Barci D G, Bollini C G, Oxman L E and Rocca M C Non-local pseudo-differential operators *Preprint* hep-th/9606183
[8] Brunetti R, Fredenhagen K and Köhler M 1995 The microlocal spectrum condition and Wick polynomials of free fields in curved spacetimes *Preprint* DESY 95-196
Köhler M 1995 *Class. Quantum Grav.* **12** 1413
[9] Ray D B 1970 *Adv. Math.* **4** 109
Ray D B and Singer I M 1971 *Adv. Math.* **7** 145
Ray D B and Singer I M 1973 *Ann. Math.* **98** 154

[10] Gilkey P G 1984 *Invariance Theory, The Heat Equation and the Atiyah–Singer Index Theorem (Math. Lecture Series 11)* (Boston, MA: Publish or Perish)
[11] Schwarz A 1979 *Commun. Math. Phys.* **67** 1
Schwarz A and Tyupkin Yu 1984 *Nucl. Phys.* B **242** 436
Schwarz A 1987 Abstracts (Part II) *Baku Int. Topological Conf. (Baku, 1987)*
Schwarz A 1978 *Lett. Math. Phys.* **2** 247
[12] Witten E 1989 *Commun. Math. Phys.* **121** 351
[13] Dixmier J 1966 Existence de traces non normales *C.R. Acad. Sci. Paris* **262** 1107
[14] Connes A 1994 *Non-commutative Geometry* (New York: Academic)
[15] Wodzicki M 1984 *Invent. Math.* **75** 143
Wodzicki M 1987 Non-commutative residue *K-theory, Arithmetic and Geometry (Lecture Notes in Math. 1289)* (Berlin: Springer) p 320
[16] Kalau W and Walze M 1995 *J. Geom. Phys.* **16** 327
[17] Ackermann T 1995 A note on the Wodzicki residue *Preprint* funct-an/9506006
[18] Gracia-Bondia J M Non-commutative geometry and the standard model: an overview hep-th/9602134
Martin C P, Gracia-Bondia J M and Varilly J C The standard model as a non-commutative geometry: the low energy regime hep-th/9605001.
[19] Elizalde E 1995 *Ten Physical Applications of Spectral Zeta Functions* (Berlin: Springer)
[20] Epstein P 1903 *Math. Ann.* **56** 615
Epstein P 1907 *Math. Ann.* **65** 205
[21] Kirsten K and Elizalde E 1995 *Phys. Lett.* **365B** 72
[22] Elizalde E 1994 *J. Math. Phys.* **35** 6100
[23] Chowla S and Selberg A 1949 *Proc. Nat. Acad. Sci. USA* **35** 317
[24] Elizalde E 1989 *J. Phys. A: Math. Gen.* **22** 931
Elizalde E and Romeo A 1989 *Phys. Rev.* D **40** 436
Elizalde E 1990 *J. Math. Phys.* **31** 170
[25] Elizalde E 1994 *J. Math. Phys.* **35** 3308
[26] Iyanaga S and Kawada Y (eds) 1977 *Encyclopedic Dictionary of Mathematics* vol II (Cambridge, MA: MIT) p 1372 ff